1.1 a)

A	B	$A \wedge B$	$\overline{A \wedge B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \vee \bar{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

b)

A	B	$A \Rightarrow B$	$\bar{A}$	$B \vee \bar{A}$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

1.2 a)

A	B	$\bar{B}$	$A \Rightarrow B$	$A \Rightarrow \bar{B}$	$(A \Rightarrow B) \wedge (A \Rightarrow \bar{B})$	$\bar{A}$	$(A \Rightarrow B) \wedge (A \Rightarrow \bar{B}) \Rightarrow \bar{A}$
1	1	0	1	0	0	0	1
1	0	1	0	1	0	0	1
0	1	0	1	1	1	1	1
0	0	1	1	1	1	1	1

b)

A	B	$\bar{A}$	$\bar{B}$	$\bar{B} \Rightarrow \bar{A}$	$A \wedge (\bar{B} \Rightarrow \bar{A})$	$A \wedge (\bar{B} \Rightarrow \bar{A}) \Rightarrow B$
1	1	0	0	1	1	1
1	0	0	1	0	0	1
0	1	1	0	1	0	1
0	0	1	1	1	0	1

1.3 Angenommen $A \Rightarrow B$ wäre folgendermaßen definiert

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	0
0	0	0

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$\overline{B}$	$\overline{A}$	$\overline{B} \Rightarrow \overline{A}$
1	1	1	1	0	0	0
1	0	0	0	1	0	0
0	1	0	0	0	1	0
0	0	0	0	1	1	1

Dann würde gelten: $A \Rightarrow B = B \Rightarrow A \quad \overline{B} \Rightarrow \overline{A} \neq A \Rightarrow B$

Angenommen $A \Rightarrow B$ wäre folgendermaßen definiert

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	0

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$\overline{B}$	$\overline{A}$	$\overline{B} \Rightarrow \overline{A}$
1	1	1	1	0	0	0
1	0	0	1	1	0	0
0	1	1	0	0	1	1
0	0	0	0	1	1	1

Dann würde gelten: $\overline{B} \Rightarrow \overline{A} \neq A \Rightarrow B$

Angenommen $A \Rightarrow B$ wäre folgendermaßen definiert

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	0
0	0	1

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$\overline{B}$	$\overline{A}$	$\overline{B} \Rightarrow \overline{A}$
1	1	1	1	0	0	1
1	0	0	0	1	0	0
0	1	0	0	0	1	0
0	0	1	1	1	1	1

Dann würde gelten: $A \Rightarrow B = B \Rightarrow A$

1.4 a)

$$\overline{A \cup B} = \{ x \mid x \notin A \cup B \} = \{ x \mid \overline{x \in A \cup B} \} = \{ x \mid \overline{x \in A \vee x \in B} \}$$

$$= \{ x \mid \overline{x \in A} \wedge \overline{x \in B} \} = \{ x \mid x \notin A \wedge x \notin B \} = \{ x \mid x \in \overline{A} \wedge x \in \overline{B} \} = \overline{A} \cap \overline{B}$$

b)

$$(A \cup B) \cap \overline{A} = (A \cap \overline{A}) \cup (B \cap \overline{A}) = \{ \} \cup (B \cap \overline{A}) = B \cap \overline{A}$$

1.5

$$(A \cup B) \cap (\overline{A} \cap \overline{B}) = (A \cup B) \cap (\overline{A \cup B}) = \{ \}$$

1.6 a)

$$\frac{\frac{x}{y} - \frac{y}{x}}{\frac{1}{x} + \frac{1}{y}} = \frac{\frac{x^2 - y^2}{xy}}{\frac{y + x}{xy}} = \frac{x^2 - y^2}{x + y} = \frac{(x+y)(x-y)}{x+y} = x - y$$

b) $$\frac{\frac{x+y}{x-y}-\frac{x-y}{x+y}}{\frac{x-y}{x+y}+\frac{x+y}{x-y}} = \frac{\frac{(x+y)^2-(x-y)^2}{(x+y)(x-y)}}{\frac{(x-y)^2+(x+y)^2}{(x+y)(x-y)}} = \frac{(x+y)^2-(x-y)^2}{(x+y)^2+(x-y)^2}$$

$$= \frac{4xy}{2x^2+2y^2} = \frac{2xy}{x^2+y^2}$$

c) $$\frac{\frac{t^{q+1}}{t^{p-2}}}{\frac{t^{1+q-p}}{t^3}} = \frac{t^{q+1}\,t^{-p+2}}{t^{1+q-p}\,t^{-3}} = \frac{t^{q-p+3}}{t^{q-p-2}} = t^{q-p+3}\,t^{-q+p+2}$$

$$= t^{q-p+3-q+p+2} = t^5$$

d) $$\frac{a^3+3a^2b+3ab^2+b^3}{a^3+2a^2b+ab^2} = \frac{a^3+3a^2b+3ab^2+b^3}{a(a^2+2ab+b^2)} = \frac{(a+b)^3}{a(a+b)^2} = \frac{a+b}{a}$$

$$= 1+\frac{b}{a}$$

1.7 a) $A(1)$: $(1+x)^1 \geq 1+1\cdot x$ stimmt

Für $x=-1$ erhält man $0^n \geq 1-n$ d.h. $1-n \leq 0$

$1-n \leq 0$ stimmt für alle $n \in \mathbb{N}$

Für $x>-1$ ist $1+x>0$ und man erhält

$A(n)$: $(1+x)^n \geq 1+nx \quad \big| \cdot (1+x)$

$\Rightarrow \quad (1+x)^n(1+x) \geq (1+nx)(1+x)$

$\Rightarrow \quad (1+x)^{n+1} \geq 1+(n+1)x+nx^2$

Wegen $nx^2 \geq 0$ ist $1+(n+1)x+nx^2 \geq 1+(n+1)x$

$\Rightarrow \quad (1+x)^{n+1} \geq 1+(n+1)x+nx^2 \geq 1+(n+1)x$

$\Rightarrow \quad (1+x)^{n+1} \geq 1+(n+1)x \qquad A(n+1)$

Diese Ungleichung stellt $A(n+1)$ dar

b)

$$A(1): \quad \sum_{k=0}^{0} q^k = 1 = \frac{1-q^1}{1-q} \quad \text{stimmt}$$

$$A(n): \quad \sum_{k=0}^{n-1} q^k = 1+q+q^2+\ldots+q^{n-1} = \frac{1-q^n}{1-q} \quad \Big| + q^n$$

$$\Rightarrow 1+q+q^2+\ldots+q^{n-1}+q^n = \frac{1-q^n}{1-q} + q^n$$

$$\Rightarrow \sum_{k=0}^{n} q^k = \frac{1-q^n}{1-q} + \frac{q^n(1-q)}{1-q} = \frac{1-q^n+q^n(1-q)}{1-q} = \frac{1-q^{n+1}}{1-q}$$

Diese Gleichung stellt $A(n+1)$ dar

c)

$$A(1): \quad \sum_{k=1}^{1} k^2 = 1 = \frac{1}{6}1(1+1)(2\cdot 1+1) \quad \text{stimmt}$$

$$A(n): \quad \sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1) \quad \Big| + (n+1)^2$$

$$\Rightarrow 1^2+2^2+\ldots+n^2+(n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

$$\Rightarrow \sum_{k=1}^{n+1} k^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

$$= \frac{1}{6}(n+1)\left[n(2n+1)+6(n+1)\right] = \frac{1}{6}(n+1)(2n^2+7n+6)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3) = \frac{1}{6}(n+1)(n+1+1)(2(n+1)+1)$$

Diese Gleichung stellt $A(n+1)$ dar

d)

$$A(1): \sum_{k=1}^{1} k^3 = 1 = \binom{1+1}{2}^2 \quad \text{stimmt}$$

$$A(n): \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \binom{n+1}{2}^2 = \left(\frac{(n+1)!}{2!(n-1)!}\right)^2 = \left(\frac{1}{2}\, n(n+1)\right)^2$$

$$= \frac{1}{4} n^2 (n+1)^2 \quad \Big| + (n+1)^3$$

$$\Rightarrow 1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \frac{1}{4} n^2 (n+1)^2 + (n+1)^3$$

$$\Rightarrow \sum_{k=1}^{n+1} k^3 = \frac{1}{4} n^2 (n+1)^2 + (n+1)^3 = \frac{1}{4}(n+1)^2 \left(n^2 + 4(n+1)\right)$$

$$= \frac{1}{4}(n+1)^2 (n^2 + 4n + 4) = \frac{1}{4}(n+1)^2 (n+2)^2$$

$$= \frac{1}{4}(n+1)^2 (n+1 + 1)^2$$

Diese Gleichung stellt $A(n+1)$ dar

e)

$$A(1): \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2} = \frac{1}{1 \cdot (1+1)} \quad \text{stimmt}$$

$$A(n): \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1} \quad \Big| + \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow \sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2} = \frac{n+1}{n+1 + 1}$$

Diese Gleichung stellt $A(n+1)$ dar

f)

$$A(1): \sum_{k=1}^{1} \frac{k}{2^k} = \frac{1}{2} = 2 - \frac{1+2}{2} \quad \text{stimmt}$$

$$A(n): \sum_{k=1}^{n} \frac{k}{2^k} = \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \quad \Big| + \frac{n+1}{2^{n+1}}$$

$$\Rightarrow \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}}$$

$$\Rightarrow \sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{2(n+2)}{2^{n+1}} + \frac{n+1}{2^{n+1}}$$

$$= 2 - \left(\frac{2(n+2)}{2^{n+1}} - \frac{n+1}{2^{n+1}} \right) = 2 - \frac{2(n+2) - (n+1)}{2^{n+1}}$$

$$= 2 - \frac{n+3}{2^{n+1}} = 2 - \frac{n+1+2}{2^{n+1}}$$

Diese Gleichung stellt $A(n+1)$ dar

g)

$$\left.\begin{aligned} &A(1):\ \sum_{k=1}^{1} k p^k = p \\ &\frac{p(1-p)-p^2(1-p)}{(1-p)^2} = \frac{(1-p)(p-p^2)}{(1-p)^2} = \frac{(1-p)^2 p}{(1-p)^2} = p \end{aligned}\right\} \Rightarrow A(1) \text{ stimmt}$$

$$A(n):\ \sum_{k=1}^{n} k p^k = \frac{p(1-p^n) - n p^{n+1}(1-p)}{(1-p)^2} \quad \Big| + (n+1)p^{n+1}$$

$$\Rightarrow \sum_{k=1}^{n} k p^k + (n+1)p^{n+1} = \frac{p(1-p^n) - n p^{n+1}(1-p)}{(1-p)^2} + (n+1)p^{n+1}$$

$$\Rightarrow \sum_{k=1}^{n+1} k p^k = \frac{p(1-p^n) - n p^{n+1}(1-p)}{(1-p)^2} + \frac{(n+1)p^{n+1}(1-p)^2}{(1-p)^2}$$

$$= \frac{p(1-p^n) - n p^{n+1}(1-p) + (n+1)p^{n+1}(1-p)^2}{(1-p)^2}$$

Zähler: $p(1-p^n) - n p^{n+1}(1-p) + (n+1)p^{n+1}(1-p)^2$

$$= p - p^{n+1} - n p^{n+1} + n p^{n+2} + (n+1)p^{n+1}(1-2p+p^2)$$

$$= p - p^{n+1} - n p^{n+1} + n p^{n+2} + (n+1)p^{n+1} - 2(n+1)p^{n+2} + (n+1)p^{n+3}$$

$$= p - p^{n+2} - (n+1)p^{n+2} + (n+1)p^{n+3}$$

$$= p(1-p^{n+1}) + (n+1)p^{n+2}(1-p)$$

$$\Rightarrow \sum_{k=1}^{n+1} k p^k = \frac{p(1-p^{n+1}) + (n+1)p^{n+2}(1-p)}{(1-p)^2}$$

Diese Gleichung stellt $A(n+1)$ dar

alternativ:

$$s_n = \sum_{k=1}^{n} k p^k = p + 2p^2 + 3p^3 + \dots + (n-1)p^{n-1} + np^n$$

$$p \cdot s_n = p^2 + 2p^3 + \dots + (n-1)p^n + np^{n+1}$$

$$s_n - p s_n = p + p^2 + p^3 + \dots + p^n - np^{n+1}$$

$$= p(1 + p + p^2 + \dots + p^{n-1}) - np^{n+1}$$

$$= p\,\frac{1-p^n}{1-p} - np^{n+1}$$

$$(1-p)s_n = p\,\frac{1-p^n}{1-p} - np^{n+1}$$

$$(1-p)^2 s_n = p(1-p^n) - np^{n+1}(1-p)$$

$$s_n = \frac{p(1-p^n) - np^{n+1}(1-p)}{(1-p)^2}$$

1.8

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{n!} = 1 \qquad \binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{(n-1)!\,n}{(n-1)!} = n$$

$$\binom{n}{n-1} = \frac{n!}{(n-1)!\,(n-(n-1))!} = \frac{(n-1)!\,n}{(n-1)!\,1!} = n$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!\,k}{(k-1)!\,k\,(n-k+1)!} + \frac{n!(n-k+1)}{k!\,(n-k)!\,(n-k+1)}$$

$$= \frac{n!\,k}{k!\,(n-k+1)!} + \frac{n!\,(n-k+1)}{k!\,(n-k+1)!} = \frac{n!\,k + n!\,(n-k+1)}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(k+n-k+1)}{k!\,(n+1-k)!} = \frac{n!\,(n+1)}{k!\,(n+1-k)!} = \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$